Hopf instantons, Chern-Simons vortices and Heisenberg ferromagnets

2001 J. Phys. A: Math. Gen. 34 3079

(http://iopscience.iop.org/0305-4470/34/14/313)

J. Phys. A: Math. Gen. **34** (2001) 3079–3082 www.iop.org/Journals/ja PII: S0305-4470(01)19876-8

Hopf instantons, Chern–Simons vortices and Heisenberg ferromagnets

P A Horváthy

Laboratoire de Mathématiques et de Physique Théorique, Université de Tours, Parc de Grandmont, F-37200 Tours, France

E-mail: horvathy@univ-tours.fr

Received 29 November 2000

Abstract

The dimensional reduction of the three-dimensional model (related to Hopf maps) of Adam *et al* is shown to be equivalent to either (i) the static, fixed-chirality sector of the non-relativistic spinor-Chern–Simons theory in 2 + 1 dimensions or (ii) a particular Heisenberg ferromagnet in the plane.

PACS numbers: 1110L, 0365, 1115

1. Scalar Chern–Simons vortices and Hopf instantons

In the non-relativistic Chern–Simons model of Jackiw and Pi [1], one considers a scalar field Ψ which satisfies a second-order nonlinear Schrödinger equation,

$$\mathrm{i}D_t\Psi = \frac{D_i D^i}{2m}\Psi - g|\Psi|^2\Psi = 0 \tag{1.1}$$

while the dynamics of the gauge field is governed by the Chern–Simons field/current identities. When the coefficient g is minus or plus the inverse of the Chern–Simons coupling constant κ, static solutions arise by solving instead the self-duality equations,

$$D_\pm\Psi \equiv (D_1 \pm \mathrm{i}D_2)\Psi = 0 \qquad (D_k = \partial_k - \mathrm{i}A_k) \tag{1.2}$$

supplemented with one of the Chern–Simons equations, namely

$$\kappa B \equiv \kappa\epsilon^{ij}\partial_i A^j = -\varrho \tag{1.3}$$

where $\varrho = \Psi^*\Psi$ is the particle density. Expressing the gauge potential from (1.2) one finds that the other Chern–Simons equations, $\kappa E^i \equiv -\kappa\big(\partial_i A^0 + \partial_t A^i\big) = \epsilon^{ij}J^j$, merely fix A_t. Then, inserting into (1.3) yields the Liouville equation, whose well known solutions provide us with Chern–Simons vortices which carry electric and magnetic fields. The self-dual solutions represent furthermore the absolute minima of the energy, cf [1].

In a recent paper, Adam *et al* [2] consider instead a massless two-spinor $\Phi = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$ on ordinary three-space, coupled to a (Euclidean) Chern–Simons field. Their field equations read

$$D_i \sigma_i \Phi = 0 \tag{1.4}$$

$$\Phi^\dagger \sigma_i \Phi = B_i. \tag{1.5}$$

Note that this model only contains a (three-dimensional) magnetic and no electric field. These authors also mention that assuming independence of x_3 and setting $A_3 = 0$, their model will reduce to the planar self-dual Jackiw–Pi system, (1.2), (1.3). The third component of (1.5) requires, in fact,

$$|\Phi_+|^2 - |\Phi_-|^2 = B. \tag{1.6}$$

The two other components imply, however, that either Φ_+ or Φ_- has to vanish. Therefore, the reduced equations read finally as one or the other of

$$D_\pm \Phi_\mp = 0 \qquad B = \pm|\Phi_\pm|^2 \qquad \text{and} \qquad \Phi_\mp = 0. \tag{1.7}$$

Fixing up the sign problem by including a Chern–Simons coupling constant κ, these equations look indeed *formally* the same as in the self-dual Jackiw–Pi case. They have, however, a slightly different interpretation: they are purely magnetic, while those of Jackiw and Pi have a non-vanishing electric field. Let us underline that the equations (1.7) differ from the second-order field equation (1.1).

2. Spinor vortices

Here we point out that the model of Adam *et al* reduces rather more naturally to a particular case of our spinor model in 2 + 1 dimensions [3]. In this theory, the four-component Dirac spinor with components Φ_-, χ_-, χ_+ and Φ_+ satisfies the Lévy-Leblond equations [4]

$$\begin{aligned} &(\vec{\sigma} \cdot \vec{D})\, \Phi + 2m\, \chi = 0 \\ &D_t\, \Phi + \mathrm{i}(\vec{\sigma}\vec{D})\, \chi = 0 \end{aligned} \tag{2.1}$$

where Φ and χ are two-component 'Pauli' spinors $\Phi = \begin{pmatrix} \Phi_- \\ \Phi_+ \end{pmatrix}$ and $\chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix}$. This non-relativistic Dirac-type equation is completed with the Chern–Simons equations

$$\begin{aligned} &B = (-1/\kappa)\big(|\Phi_+|^2 + |\Phi_-|^2\big) \\ &E_i = (1/\kappa)\epsilon_{ij} J_j \qquad J_j = \mathrm{i}\big(\Phi^\dagger \sigma_j \chi - \chi^\dagger \sigma_j \Phi\big). \end{aligned} \tag{2.2}$$

In the static and purely magnetic case, $A_t = 0$, and choosing $\chi_+ = \chi_- = 0$, the second equation in (2.2) is identically satisfied, leaving us with the coupled system

$$\begin{aligned} &D_+ \Phi_- = 0 \\ &D_- \Phi_+ = 0 \\ &B = (-1/\kappa)\big(|\Phi_+|^2 + |\Phi_-|^2\big). \end{aligned} \tag{2.3}$$

Choosing a fixed chirality, $\Phi_- \equiv 0$ or $\Phi_+ \equiv 0$, yields furthermore one of the two systems

$$\begin{aligned} &D_\pm \Phi_\mp = 0 \\ &B = (-1/\kappa)\, |\Phi_\pm|^2 \end{aligned} \tag{2.4}$$

which, for $\kappa = 1$, are precisely (1.7). For both signs we obtain the Liouville equation; regular solutions were obtained for Φ_+ when $\kappa < 0$, and for Φ_- when $\kappa > 0$. They are again purely magnetic, and carry non-zero spin[1].

[1] The same self-dual equations arise in the relativistic model of [3].

It would be easy keep both terms in (1.7) by allowing a non-vanishing (but still x_3-independent) A_3. Then one would lose the equations $D_\pm\Phi_\mp = 0$, however. The impossibility of having both components in (2.3) but not in (1.7) comes from the type of reduction performed: while for spinors one eliminates non-relativistic time, (1.7) originates from a spacelike reduction. The difference is also related to the structure of the Lévy-Leblond equation (2.1), which can be obtained by *lightlike* reduction from a massless Dirac equation in four dimensions, while (1.4) arises by spacelike reduction [3].

It is interesting to observe that eliminating χ in favour of Φ in the Lévy-Leblond equation (2.3) yields

$$\mathrm{i}D_t\Phi = \left[-\frac{1}{2m}D_iD^i + \frac{1}{2m\kappa}\left(|\Phi_+|^2 + |\Phi_-|^2\right)\sigma_3\right]\Phi. \tag{2.5}$$

For both chiralities, we obtain hence a second-order equation of the Jackiw–Pi form (1.1), but with opposite signs, i.e. with attractive/repulsive coupling.

It is worth noting that the minima of the energy correspond to the coupled equations (2.3) and *not* to (2.4). In fact, the identity

$$|\vec{D}\Phi|^2 = |D_+\Phi_-|^2 + |D_-\Phi_+|^2 - \frac{1}{2m\kappa}|\Phi|^2\Phi^\dagger\sigma_3\Phi + \text{ surface terms}$$

shows that the energy of a field configuration,

$$H = \int\left\{\frac{1}{2m}|\vec{D}\Phi|^2 + \frac{1}{2m\kappa}|\Phi|^2\Phi^\dagger\sigma_3\Phi\right\}\mathrm{d}^2\vec{x}$$

is actually

$$H = \frac{1}{2m}\int \mathrm{d}^2\vec{r}\,\left\{|D_+\Phi_-|^2 + |D_-\Phi_+|^2\right\} \tag{2.6}$$

which is positive definite, $H \geqslant 0$, provided the currents vanish at infinity. The 'Bogomolny' bound is furthermore saturated precisely when (2.3) holds. Its solutions are therefore stable. Hence, it is (2.3) that should be considered as the true self-duality condition.

3. Heisenberg ferromagnets

The relative minus sign of the component densities in the 'provisional' formula (1.6) differs from ours in (2.3), and is rather that in the two-dimensional Heisenberg model studied by Martina *et al* [5]. Here the spin, represented by a unit vector $\boldsymbol{S}$, satisfies the Landau–Lifschitz equation $\partial_t\boldsymbol{S} = \boldsymbol{S}\times\triangle\boldsymbol{S}$. In the so-called tangent-space representation, $\boldsymbol{S}$ is replaced by two complex fields, Ψ_+ and Ψ_-, each of which satisfies a (second-order) nonlinear Schrödinger equation,

$$\mathrm{i}D_t\Psi_\pm = -\left[D_iD^i + 8|\Psi_\pm|^2\right]\Psi_\pm \tag{3.1}$$

as well as a geometric constraint, $D_+\Psi_- = D_-\Psi_+$. The covariant derivatives here refer to a Chern–Simons-type abelian gauge field,

$$\begin{aligned}&B = -8\left(|\Psi_+|^2 - |\Psi_-|^2\right)\\ &E_i = 8\epsilon_{ij}J_j \qquad J_i = \left(\Psi_+^*D_i\Psi_+ - \Psi_+(D_i\Psi_+)^*\right) - \left(\Psi_-^*D_i\Psi_- - \Psi_-(D_i\Psi_-)^*\right).\end{aligned} \tag{3.2}$$

It is now easy to check that in the static and purely magnetic case, these equations can be solved by the first-order coupled system

$$\begin{aligned}&D_\pm\Psi_\mp = 0\\ &B = -8\left(|\Psi_+|^2 - |\Psi_-|^2\right).\end{aligned} \tag{3.3}$$

For $\Psi_+ \equiv 0$ or $\Psi_- \equiv 0$, we obtain once again the equation of Adam *et al*. In the general case, (3.3) leads to an interesting generalization of the Liouville equation: making use of its conformal properties, Martina *et al* have shown that it can be transformed into the 'sinh–Gordon' form

$$\triangle\sigma = -\sinh\sigma \tag{3.4}$$

where σ is suitably defined from Ψ_+ and Ψ_-. Although this equation has no finite-energy regular solution defined over the whole plane [6], it admits doubly periodic solutions, i.e. solutions defined in cells with periodic boundary conditions [7]. This generalizes the results of Olesen [8] in the scalar case. A similar calculation applied to the general SD equations, (2.3), of our spinor model would yield

$$\triangle\sigma = -\cosh\sigma \tag{3.5}$$

whose (doubly periodic) solutions could be interpreted as nonlinear superpositions of the chiral vortices in [3].

References

[1] Jackiw R and Pi S-Y 1990 *Phys. Rev. Lett.* **64** 2969
Jackiw R and Pi S-Y 1990 *Phys. Rev.* D **42** 3500
For reviews, see, e.g., Jackiw R and Pi S-Y 1992 *Prog. Theor. Phys. Suppl.* **107** 1
Dunne G 1995 Self-Dual Chern–Simons Solitons *Springer Lecture Notes in Physics vol 36* (Berlin: Springer)
[2] Adam C, Muratori B and Nash C 2000 *Phys. Lett.* B **479** 329
[3] Duval C, Horváthy P A and Palla L 1995 *Phys. Rev.* D **52** 4700
Duval C, Horváthy P A and Palla L 1996 *Ann. Phys., NY* **249** 265
Cho Y M, Kim J W and Park D H 1992 *Phys. Rev.* D **45** 3802
[4] Lévy-Leblond J-M 1967 *Commun. Math. Phys.* **6** 286
[5] Martina L, Pashaev O K and Soliani G 1993 *Phys. Rev.* B **48** 15787
[6] Dunne G, Jackiw R, Pi S-Y and Trugenberger 1991 *Phys. Rev.* D **43** 1332
[7] Ting A C, Chen H H and Lee Y C 1984 *Phys. Rev. Lett.* **53** 1348
Ting A C, Chen H H and Lee Y C 1987 *Physica* D **26** 37
[8] Olesen P 1991 *Phys. Lett.* B **265** 361
Olesen P 1991 *Phys. Lett.* B **268** 389